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Evaluating Utility-Based and Coherent Risk Measures in Financial Risk Management

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Abstract: *Risk is an inherent part of our daily personal and professional life, and it can be found in every aspect of it. Particularly in finance and economics, managing and understanding risk is very important, yet defining and measuring it is a challenge due to its subjective nature.*

Effective risk management requires different tools and methodologies, many of which originated from Markowitz's work in 1952. This paper examines risk measures, emphasizing the concept of coherence introduced by Artzner et al. (1999). A coherent risk measure satisfies monotonicity, positive homogeneity, sub-additivity, and translation invariance.

Key measures, including variance, skewness, Value at Risk (VaR), and Conditional Tail Expectation (CTE) will be analyzed in this paper. While widely used, variance and skewness lack coherence. VaR, popular in finance, also fails to meet coherence standards. In contrast, CTE emerges as a coherent and reliable metric, addressing VaR's shortcomings by focusing on extreme scenarios.

1. INTRODUCTION

Risk is and always has been a concern in our lives. However, risk is not an objective concept, thus, it is not easy to give a precise definition to it. Roughly speaking, risk means a chance that a certain injury or loss associated with a given action happens. Because of the great variety of risks and the increased coincidence of different events associated with risk occurring, it is becoming increasingly difficult to identify and measure risks. Also, it is becoming indispensable for one working in certain areas of finance, insurance, microeconomics, etc. to be able to measure, control and mitigate risk. Because of the importance of risk management, a great deal of studies, tools and measures have been developed in the past decades, highlighting the need for a proper definition of risk as well as specific measures to calculate and predict risks. In this paper, we will show and describe different methods and measures which are used to quantify risk. Since the pioneering work of Markowitz (1952), plenty of different risk measures and methods have been proposed. Acerbi and Tasche (2002) state that if a measure is not coherent, then it can simply not be named a risk measure, while the notion of coherence itself was first introduced by Artzner et al. (1999) and currently, is a fundamental concept related to the acceptability of a risk measure.

2. COHERENCE OF A RISK MEASURE

In order to discuss the coherence of a risk measure, some preliminary details are needed. First, let (Ω, F, P) be a probability space and let $L^0(\Omega, F, P)$ (or short L^0) denote the space of all random variables (i.e. all measurable functions) on this probability space. A financial position X is an element of $L^0(\Omega, F, P)$ modeling an uncertain payoff. A risk measure on the other hand is a function

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$\rho : L^0 \rightarrow R$ (or sometimes $\rho : L^0 \rightarrow R \cup \{+\infty\}$) which defines a preference order on L^0 allowing a decision-maker to choose between any two given positions (Assa, 2011).

In easier terms, a coherent risk measure means the risk measure covers all the maximum risks possible. On the other hand, four mathematical axioms must be fulfilled for a risk measure to be called coherent (Artzner et al., 1999).

Definition 1. Let K be a convex cone[3] in L^0 containing R (R as the space of constant functions). A function $\rho : K \rightarrow R$ is a coherent risk measure if ρ is

1. positive homogeneous, i.e. $\rho(\lambda X) = \lambda\rho(X)$, $\forall X \in K$ and $\lambda \in (0,+\infty)$.
2. sub-additive, i.e. $\rho(X+Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y \in K$.
3. translation invariant, i.e. $\rho(X+m) = \rho(X) - m$, $\forall X \in K$ and $m \in R$.
4. decreasing, i.e. $\rho(X) \leq \rho(Y)$, $\forall X, Y \in K$ such that $X \geq Y$ almost surely.

If axioms (1) and (2) are replaced by

5. convexity, i.e. $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\forall X, Y \in K$ and $\lambda \in [0,1]$, the risk measure ρ is called a convex risk measure.

In economic terms, Axiom (1) shows that increasing exposure to a risky position leads to a proportional rise in the level of risk. Notably, this axiom introduces a new preference relation in the framework of coherent risk measures that cannot be replicated within the expected utility approach (Assa, 2011). Axiom (2) reflects the widely accepted principle that diversification reduces risk, a feature also present in the expected utility framework due to the concavity of the utility function. Axiom (3) grants coherent risk measures a cash-invariance property[4]. Lastly, Axiom (4) shows that if the payoff of one financial position consistently exceeds that of another, the risk measure will preserve this order (Assa, 2011). Based on these axiom criteria or properties some risk measures will be described and their coherence (or lack of) will be proved.

3. VARIANCE

Variance is a statistical measure that quantifies the dispersion of returns around the mean (average) return of an investment or financial position. In economic terms, it is used as a risk measure to represent the uncertainty or variability in the potential outcomes of an asset's returns. A higher variance indicates greater variability and, therefore, a higher level of risk, as it implies a wider range of possible outcomes around the expected return (Wasserman, 2013).

However, variance as a risk measure assumes that all deviations from the mean—both positive and negative—are equally undesirable, which may not align with how investors perceive risk (Kagan & Shepp, 1998). For this reason, while variance is a foundational concept in finance (e.g., in portfolio theory), it is often supplemented by other risk measures, such as standard deviation, Value at Risk (VaR), or downside risk measures, to better capture real-world preferences and risk aversion (Grootveld & Hallerbach, 1999).

[3] In linear algebra, a cone is a subset of a vector space that is closed under positive scalar multiplication; that is, Cis a cone if $x \in C$ implies $sx \in C$ for every positive scalar s.

[4] The cash-invariance property ensures that the risk measure responds linearly and predictably to changes in cash levels, which is crucial for practical applications like setting capital reserves or determining the amount needed to hedge a risky position. It reflects the idea that holding additional cash reduces financial risk, as cash serves as a buffer against potential losses.

3.1. Definition

Let X be a numerically valued random variable with expected value $\mu = E(X)$. Then the variance of X, denoted by $Var\,(X)$, is given as

$$Var\,(X) = E((X - \mu)^2) \tag{1}$$

3.2. Coherence of Variance

As mentioned above a measure cannot be called a risk measure, in case it is not coherent. And to show that $Var(X)$ is not coherent, it is needed to show that $Var(X)$ is neither positive homogenous nor sub-additive.

Let X and Y, be two random variables. Let $Var(X)$ indicate the variance of X, and with $\sigma(X)$ its standard deviation. It is known that $\sigma(X) = (VarX)^{1/2}$

1. Positive homogeneity: If X is multiplied by a scalar α (i.e. a number), the properties of variance tell us that $Var(\alpha X) = \alpha^2 Var(X)$, because the variance is not linear. However, $\alpha^2\, Var(X) \neq \alpha Var(X)$, hence the variance is **not** positive homogenous.

Proof: Let X be a random variable and α, a constant. Then $Var(\alpha X) = \alpha^2 Var(X)$. From the definition of variance and repeatedly applying linearity of expectation, we have:

$$\begin{aligned} Var(\alpha X) &:= E((\alpha \mathrm{X} - \mathrm{E}\,[\alpha \mathrm{X}])^2) \\ &= E[(\alpha \mathrm{X})^2 - 2\alpha \mathrm{X}\,\mathrm{E}\,[\mathrm{aX}] + E^2[\alpha \mathrm{X}] \\ &= E[(\alpha \mathrm{X})^2] - E\,[2\alpha X\, E\,[\alpha X]] + E^2[\alpha X \\ &= \alpha^2 E[X^2] - 2\,E\,[\alpha X]\,E\,[\alpha X] + E^2\,[\alpha X] \\ &= \alpha^2 E[X^2] - \alpha^2 E^2\,[X] = \alpha^2 (E\,[X]^2 - E^2\,[X]) \\ &= a^2\,Var\,X^2 \end{aligned} \tag{2}$$

2. Sub-additivity: From the properties of variance, it is known that

   $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$,

   where $Cov(X,Y)$ is the so-called covariance[5].

   Using correlation $\rho(X,Y) = cov(X,Y)/(\sigma(X)\sigma(Y))$, the $Var(\mathrm{X} + Y)$ can be re-written as :

$$Var(X + Y) = Var(X) + Var(Y) + 2\rho(X,Y)\sigma(X)\sigma(Y) \tag{3}$$

Unless $\rho(X,Y)$ is zero (X and Y are not linearly dependent) or negative (they are negatively correlated), we have that the right-hand side of the previous equation is always bigger than the simple sum of the variances, therefore $Var(X + Y) \geq Var(X) + Var(Y)$. Therefore, variance is not sub-additive. Even if the other two properties (monotonicity and translation invariance) are respected, the variance is not a coherent risk measure.

[5] Covariance is a measure of the joint variability of two random variables. The sign of the covariance, therefore, shows the tendency in the linear relationship between the variables (Rice, 2007). In economic and financial terms, covariance shows how two assets move in relation to each other. It provides diversification and reduces the overall volatility for a portfolio. A positive covariance indicates that two assets move in tandem. A negative covariance indicates that two assets move in opposite directions. In the construction of a portfolio, it is important to attempt to reduce the overall risk and volatility while striving for a positive rate of return.

4. SKEWNESS

Another measure of risk is its third moment, differently known as the skewness. In probability theory and statistics, skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean (Groeneveld & Meeden, 1984). Skewness not only measures asymmetry in distribution but also provides insights into the nature of extreme values or outliers. It quantifies how much the distribution of data deviates from a symmetrical bell curve. When the skewness is positive, it often suggests that the mean is greater than the median due to the influence of high outliers. Conversely, negative skewness implies that the mean is less than the median, often because of the impact of low outliers (Von Hippel, 2005).

4.1. Definition

Mathematically, the skewness of a random variable X is the third standardized moment γ_1, defined as

$$\gamma_1(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3} = \frac{E[(X-\mu)^3]}{(E[(X-\mu)^2])^{3/2}} \tag{3}$$

where μ is the mean, σ is the standard deviation, E is the expectation operator and μ_3[6] is the third central moment. Generally, skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point.

4.2. Types of Skewness

The first thing noticeable about a distribution's shape is whether it has one mode (peak) or more than one. If it's unimodal, the next thing is noticeable whether it's symmetric or skewed to one side, just as seen in Figure 1 (Doane & Seward, 2011).

1. If the bulk of the data is at the left and the right tail is longer, the distribution is skewed right or positively skewed.
2. If the peak is toward the right and the left tail is longer, the distribution is skewed left or negatively skewed.
3. And in the case where the skewness = 0, then the data are perfectly symmetrical.

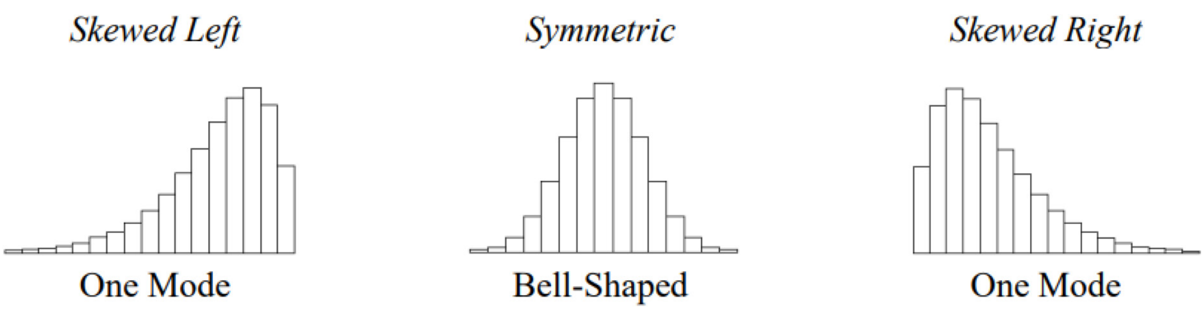

Figure 1. Types of Skewness illustrated by histograms
Source: Doane and Seward (2011)

[6] A central moment is a moment of a probability distribution of a random variable about the random variable's mean; that is, it is the expected value of a specified integer power of the deviation of the random variable from the mean (Stirzaker, 1999).

4.3. Coherence of Skewness

Skewness is also not a coherent risk measure, because it violates monotonicity, subadditivity, positive homogeneity, and translation invariance. In order not to prolong this paper, only one of these violations (which suffices to not define a risk measure as coherent) will be shown.

For $\lambda > 0$, a coherent risk measure must satisfy $\rho(\lambda X) = \lambda\rho(X)$. From the definition of skewness, we have that

$$\gamma_1(\lambda X) = \frac{E[(\lambda X - \lambda\mu)^3)]}{(\lambda\sigma)^3} = \frac{\lambda^3 E[(X-\mu)^3]}{\lambda^3\sigma^3} = \gamma_1(X). \tag{5}$$

Thus, skewness is scale-invariant (remains the same regardless of the chosen λ), which violates the positive homogeneity property of coherence.

5. VALUE AT RISK

The need to improve control of financial risks has led to a uniform measure of risk called value at risk (VaR), which the private sector is increasingly adopting as a first line of defense against financial risks (Holton, 2014). The Basel Committee on Banking Supervision announced in April 1995 that capital adequacy requirements for commercial banks are to be based on VAR[7] and in December 1995, the Securities and Exchange Commission issued a proposal that requires publicly traded U.S. corporations to disclose information about derivatives activity, with a VAR measure as one of three possible methods for making such disclosures (Jorion, 1996). Thus, the unmistakable trend is toward more-transparent financial risk reporting based on VAR measures.

5.1. Definition

Value at Risk is a statistical measure that aggregates all the risks of a portfolio into a single number suitable for use in the boardroom, reporting to regulators, or disclosure in an annual report. It is simply a way to describe the magnitude of likely losses in a portfolio, so VAR summarizes the worst expected loss over a target horizon within a given confidence interval (Duffie & Pan, 1997). More precisely, VaR is defined as follows.

Let X be a random variable and $\alpha \in [0,1]$, q is called an α-quantile if

$$P[X < q] \leq \alpha \leq P[X \leq q]. \tag{6}$$

Given the random variable X and a number $\alpha \in [0,1]$, define

$$\mathrm{VaR}_\alpha(X) = -q_\alpha(X). \tag{7}$$

X is called VaR-acceptable if $\mathrm{VaR}_\alpha(X) \leq 0$ or, equivalently, if $q_\alpha(X) \geq 0$.

VaR has the following properties:
1. $X \geq 0 \rightarrow \mathrm{VaR}_\alpha(X) \leq 0$;
2. $X \geq Y \rightarrow \mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y)$
3. $\mathrm{VaR}_\alpha(X)\ (\beta X) = \beta \mathrm{VaR}_\alpha(X)\ (X)$, for $\forall \beta \geq 0$; and
4. $\mathrm{VaR}_\alpha(X)\ \ (X + k) = \mathrm{VaR}_\alpha(X)\ (\mathrm{X}) - k$, for $\forall k \in R$.

[7] Also known as BCBS, is a committee of banking supervisory authorities that was established by the central bank governors of the Group of Ten (G10) countries in 1974.

VaR is always specified with a given confidence level α – typically $\alpha = 95\%$ or $\alpha = 99\%$. This generalized definition of VaR can be calculated for any random variable.

5.2. Coherence of Value at Risk

Consider two independent random variables X and Y (representing portfolio returns) and define Value at Risk as follows:

$$\text{VaR}_\alpha = \inf \{x \in R : P(Z \leq x) \leq 1 - \alpha\} \tag{8}$$

Where Z represents a portfolio's return, and α ∈ (0,1) is the confidence level. The subadditivity property requires:

$$\text{VaR}_\alpha(X + Y) \leq \text{VaR}_\alpha(X) + \text{VaR}_\alpha(XY). \tag{9}$$

The subadditivity property ensures that diversification does not increase risk, reflecting the principle that combining two portfolios should not lead to higher risk than assessing them individually. In order to show that VaR fails to satisfy the coherence axioms, we need to construct a counterexample.

Let X and Y have the following probability distributions: $P(X = -1) = 0.05$, $P(\text{X} = 0) = 0.95$ and $\text{P}(Y = -1) = 0.05$, $\text{P}(Y = 0) = 0.95$. Furthermore, assume X and Y are independent random variables.

At confidence level $\alpha = 0.95$, compute $\text{VaR}_{0.95}(X)$. Since $P(X = -1) = 0.05$, the worst 5% of outcomes occur when $X = -1$. Thus: $\text{VaR}_{0.95}(X) = 1$. Similarly, $\text{VaR}_{0.95}(Y) = 1$, because Y has the same distribution as X.

Now calculate $\text{VaR}_{0.95}(X + Y)$. The possible value of $X + Y$ are: $X + Y = -2$, $X + Y = -1$, $X + Y = 0$. To determine $\text{VaR}_{0.95}(X + Y)$, the smallest value x such that $\text{P}(X + Y \leq x) \leq 0.05$ is needed. From the distribution $\text{P}(X + Y \leq -2) = 0.0025$, $\text{P}(X + Y \leq -1) = 0.0025 + 0.095 = 0.0975$. Since $\text{P}(X + Y \leq -1) > 0.05$, the worst 5% of outcomes occur at $X + Y = -1$. Thus $\text{VaR}_{0.95}(X + Y) = 1$.

The subadditivity axiom on the other hand requires that $\text{VaR}_{0.95}(X + Y) \leq \text{VaR}_{0.95}(X) + \text{VaR}_{0.95}(Y)$. From the computed values of the counterexample, we know that $\text{VaR}_{0.95}(X + Y) = 1$, $\text{VaR}_{0.95}(X) = 1$ and $\text{VaR}_{0.95}(Y) = 1$, making the satisfaction of the subadditivity axiom impossible to satisfy. This means that even though VaR is commonly used in most of the financial and banking sectors, it still does not comply with the definition of a coherent risk measure.

5.3. Drawbacks of Value at Risk

As mentioned VaR is not a coherent risk measure, meaning that there are some drawbacks in using it as an adequate measure of risk (Frey & Embrechts, 2010):

1. First, VaR loses convexity properties, so it does not fulfill the property of sub-additivity, therefore it cannot be a coherent measure.
2. It is tail insensitive. It tells us that in the $\alpha \cdot 100\%$ of the cases the loss will not be greater than a certain level, but it does not give any information about the size of the loss in the remaining $(1 - \alpha) \cdot 100\%$ of the cases.
3. It is ineffective in recognizing the dangers of concentrating on credit risk.
4. Finally, VaR does not indicate the severity of the economic consequences of exposure to rare events.

6. CONDITIONAL TAIL EXPECTATION

The quantile risk measure (meaning the VaR, discussed in Chapter 5) assesses the "worst case" loss, where the worst case is defined as the event with a $(1 - \alpha)$ probability. As we mentioned above, one problem with the quantile risk measure is that it does not take into consideration what the loss will be if that $(1 - \alpha)$ worst-case event occurs (Danielsson & de Vries, 1997). The loss distribution above the quantile (VaR) does not affect the risk measure. The Conditional Tail Expectation (or CTE) was chosen to address some of the problems with the quantile risk measure (VaR). It was proposed simultaneously by several research groups, so it has many names, including Tail Value at Risk (or Tail-VaR) or Tail Conditional Expectation (or TCE) (Peng, 2009).

TCE provides a measure of the riskiness of the tail of a distribution and is an index that has gained popularity over the years. It has been getting more and more attention for measuring risk in any situation with a non-normal distribution of losses.

6.1. Definition

Like the quantile risk measure, the CTE is defined using some confidence level α, $0 \leq \alpha \leq 1$. As with the quantile risk measure, α is typically 90%, 95% or 99%. In other words, the CTE is the expected loss given that the loss falls in the worst $(1 - \alpha)$ part of the loss distribution. The worst $(1-\alpha)$ part of the loss distribution is the part above the α-quantile, q_α. If q_α falls in a continuous part of the loss distribution (that is, not in a probability mass) then we can interpret the CTE at confidence level α, given the α-quantile risk measure q_α, as

$$CTE_\alpha = E\,[L \mid L > q_\alpha] = E\,[L \mid L > \mathrm{VaR}_\alpha(L)], \tag{10}$$

where L is the random loss and VaR is the Value at Risk. Intuitively, CTE measures the average loss exceeding the α-quantile of the distribution, considering the worst-case outcomes beyond the Value-at-Risk threshold (Artzner et al., 1999). This formula does not work if q_α falls in a probability mass, that is, if there is some $\varepsilon > 0$ such that $q_{\alpha+\varepsilon} = q_\alpha$.

6.2. Coherence of Conditional Tail Expectation

1) Monotonicity: If $X \leq Y$ almost most surely, then $\mathrm{CTE}_\alpha(X) \leq \mathrm{CTE}_\alpha(Y)$. This holds because the higher losses in Y will increase both the VAR_α and the expected losses beyond that threshold.
2) Subadditivity: $\mathrm{CTE}_\alpha(X + Y) \leq \mathrm{CTE}_\alpha(X) + \mathrm{CTE}_\alpha(Y)$. This property ensures diversification benefits, as combining risks X and Y cannot result in a greater risk than summing their individual risks. The mathematical justification arises from the fact that CTE, being an expectation conditional on a tail event, is a convex function.
3) Positive Homogeneity: For any $\lambda > 0$, $\mathrm{CTE}_\alpha(\lambda X) = \lambda \mathrm{CTE}_\alpha(X)$. Scaling all outcomes by λ scales the Value-at-Risk and the expected shortfall by the same factor.
4) Translation Invariance: For a constant c, $\mathrm{CTE}_\alpha(X + c) = \mathrm{CTE}_\alpha(X) + c$. Adding a constant c shifts all outcomes by c, which increases the expected shortfall by the same amount.

The CTE has become a very important risk measure (since it has proven to be coherent) in actuarial practice. It is intuitive, easy to understand and to apply with simulation output. It provides a more comprehensive assessment of risk compared to Value-at-Risk (VaR) by considering not just the threshold loss (VaR) but also the average of losses beyond that threshold (Artzner et al., 1999). Moreover, unlike VaR, which only provides a point estimate, CTE delivers a richer picture of potential risk, making it more reliable for stress testing and financial decision-making in volatile markets. As a mean, it is more robust concerning sampling error than the quantile (Landsman & Valdez, 2005).

7. CONCLUSION

In conclusion, the concept of coherence in risk measures is crucial for ensuring that financial risk is evaluated consistently and effectively. Coherent risk measures provide a robust framework by satisfying key properties such as subadditivity, monotonicity, positive homogeneity, and translation invariance, which are essential for rational decision-making and effective risk management. Among the discussed risk measures, Conditional Tail Expectation (CTE) stands out as a fully coherent measure, capturing tail risks comprehensively and offering a more reliable perspective for extreme loss scenarios. In contrast, Value-at-Risk (VaR), while widely used in the industry, fails to meet the subadditivity property, which limits its ability to reflect diversification benefits. Similarly, Variance, although mathematically simple and commonly employed, is not coherent due to its inability to differentiate between upside and downside risk, as it considers all deviations from the mean equally. Skewness, while informative for asymmetry, lacks coherence as it is not monotonic or sub additive, making it more suited as a complementary measure rather than a standalone risk metric.

Despite its limitations, Value-at-Risk (VaR) remains the industry standard and the most commonly used risk measure in the financial industry due to its regulatory acceptance and simplicity. However, as the industry evolves, there is growing recognition of the superior theoretical properties and practical relevance of CTE, especially for stress testing and assessing tail risks in volatile markets.

References

Acerbi, C., & Tasche, D. (2002). On the coherence of expected shortfall. *Journal of banking & finance*, *26*(7), 1487-1503.
Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, *9*(3), 203-228.
Assa, H. (2011). *On some aspects of coherent risk measures and their applications* [Doctoral dissertation, University of Copenhagen]. University of Copenhagen Research Portal.
Danielsson, J., & de Vries, C. G. (1997). *Extreme returns, tail estimation, and value-at-risk* (Vol. 11). University of Iceland, Institute of Economic Studies.
Doane, D. P., & Seward, L. E. (2011). Measuring skewness: a forgotten statistic?. *Journal of statistics education*, *19*(2).
Duffie, D., & Pan, J. (1997). An overview of value at risk. *Journal of derivative*
Frey, R., & Embrechts, P. (2010). *Quantitative Risk Management*. Princeton University Press.
Groeneveld, R. A., & Meeden, G. (1984). Measuring skewness and kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *33*(4), 391–399.
Grootveld, H., & Hallerbach, W. (1999). Variance vs downside risk: Is there really that much difference?. *European Journal of operational research*, *114*(2), 304-319.
Holton, G. A. (2014). Value-at-Risk: Theory and Practice second edition, e-book. *URL: https:// www. value-at-risk. net/.(last accessed on 2020-06-26).*
Jorion, P. (1996). Risk2: Measuring the risk in value at risk. *Financial analysts journal*, *52*(6), 47-56.
Kagan, A., & Shepp, L. A. (1998). Why the variance?. *Statistics & probability letters*, *38*(4), 329-333.
Landsman, Z., & Valdez, E. A. (2005). Tail conditional expectations for exponential dispersion models. *ASTIN Bulletin: The Journal of the IAA*, *35*(1), 189-209.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy, 60*(2), 151-158.

Peng, J. (2009, July). Value at risk and tail value at risk in uncertain environment. In *Proceedings of the 8th International Conference on Information and Management Sciences* (pp. 787-793).

Rice, J. A. (2007). *Mathematical statistics and data analysis* (Vol. 371). Thomson/Brooks/Cole.

Stirzaker, D. (1999). *Probability and random variables: a beginner's guide*. Cambridge University Press.

Von Hippel, P. T. (2005). Mean, median, and skew: Correcting a textbook rule. *Journal of Statistics Education*, *13*(2).

Wasserman, L. (2013). *All of statistics: a concise course in statistical inference*. Springer Science & Business Media.